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Representations of complex functions, means on the regular *n*-gon and applications to gravitational potential

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Abstract
We present a method to analytically compute means of functions on regular n-gons and to study cyclic quantities of the complex variable. To achieve this, we construct representations of complex functions and compact expressions of their mean based on the use of a scalar product. Applied in the field of celestial mechanics, this method leads to results concerning gravitational potential and relative equilibrium composed by nested polygons.

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Introduction

The following elementary problem is at the origin of this paper: consider the system composed by n identical masses placed at the vertices of a regular n-gon and the corresponding gravitational field.

A natural question is to determine the points M that satisfy the central property (i.e. the field generated by the previous system at M is straight towards the centre of the n-gon).

In 1924, Lindow [6] answered partially the first question (for the Newtonian potential and a generic point M in the plane of the polygon), building a nice integral formula based on the use of Laplace's coefficients.

The aim of this paper is to generalize Lindow's formula and result to a larger class of potentials and to present a powerful method to study some particular cyclic quantities of the complex variable. To achieve this, we build new representations of a particular class of complex functions (namely moduli of power series and their momenta) and use them to compute means on the regular n-gon.

To obtain the most general results, we specify the kind of potential studied only in the very last sections and then give explicit representations for homogeneous potentials (the Newtonian kind being a particular case).

First, using a scalar product, we formally express some complex functions in order to evaluate their mean on the regular n-gon in an elegant and compact way.

Then, using Cauchy's formula and some manipulations in the complex field, we give explicit representations of the scalar product used. The last sections are devoted to applications of the results in the case of a homogeneous potential and in words of relative equilibria.

Known examples

Vortices in the plane

Consider n vortices with equal vorticity $\left(\frac{1}{n}\right)$ disposed at the vertices of a regular unit n-gon and a point M in the same plane referred to by its affix z. The potential of that system at M can be written as

$$V(z) = \frac{1}{n}\sum_{j=1}^{n} \ln(|z - \omega_j|).$$

Using the exceptional structure of the logarithm, this leads *ipso facto* to the compact expression

$$V(z) = \frac{1}{n} Ln|1 - z^n|.$$

Newtonian case

Where previously we had vortices, let us now consider n bodies with equal masses $\left(\frac{1}{n}\right)$. The Newtonian potential generated at M is

$$V_1(z) = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{|z - \omega_j|}.$$

Computing a change of variable in a formula due to Lindow [4] leads to the representation

$$V_1(z) = \frac{1}{\pi}\int_0^1 \frac{1}{\sqrt{\tau(1-\tau)}} \frac{1}{1 - |z|^2\tau} \frac{1 - (|z|\tau)^{2n}}{|1 - (\tau|z|)^n|^2}\, \mathrm{d}\tau.$$

In the following sections, we show how to systematically generalize the previous representations and how to use them to get properties of some cyclic quantities.

1. Formal representations of complex functions

The following preliminary lemma is intended to justify the convergence of the series involved in the identities we are going to construct and the validity of certain manipulations on those identities.

Lemma 1. *Let $F(\tau) = \sum_{k\geqslant 0} f_k \tau^k$ (resp. $G(\tau) = \sum_{k\geqslant 0} g_k \tau^k$) be a complex series with convergence radius ρ_F (resp. ρ_G). If $\rho_F \rho_G > 1$ then*

$$F \star G = \sum_{k\geqslant 0} f_k g_k$$

is absolutely convergent. Moreover, for every non-zero complex number λ, if $F_\lambda(\tau) = F(\lambda\tau)$ and $G_\mu(\tau) = G(\mu\tau)$, then

$$F \star G = F_\lambda \star G_{\frac{1}{\lambda}}.$$

Proof. There exists r such that $\frac{1}{\rho_G} < r < \rho_F$.

$$f_k \lambda^k \frac{g_k}{\lambda^k} = f_k g_k = f_k r^k \frac{g_k}{r^k}.$$

$\frac{g_k}{r^k}$ is bounded and $f_k r^k$ is an absolutely convergent series, so we get the result. Throughout this paper, the variable τ plays the role of a mute variable and appears to make precise the considered series (for instance, conjugacy affects only the coefficients of a series in τ, never τ itself). □

Proposition 1. *Let W denote an entire series with convergence radius at least 1, and z a complex number such that $|z| < 1$. Then, the following identities hold*

$$W(z) = W(\tau) \star \frac{1}{1 - \tau z} = W(\tau|z|) \star \frac{1}{1 - \tau \frac{z}{|z|}} \tag{1}$$

and if $z \neq 0$

$$(W(z) - W(\tau|z|)) \star \overline{\frac{1}{z - \tau|z|}} = 0. \tag{2}$$

The first identity is a direct consequence of the fundamental representation:

$$w_k = W(\tau) \star \tau^k$$

where w_k may be seen as the linear image of τ^k. The second one is a consequence of the first one and of the bilinearity of the scalar product $\star$.

Remark 1. This elementary proposition, in its spirit, is eloquent and its interest is already visible: in $W(z)$, the roles of $|z|$ and $\arg(z)$ have been dissociated which is very useful when performing certain algebraic computations (in particular those which only affect the angular part of z).

Compare (2) with the Cauchy's formula for an analytic function.

In this section we will build special non-elementary representations of the previous kind.

Proposition 2. *Let F and G be series with convergence radii greater or equal to 1 and let z be so that $|z| < 1$. Then*

$$\overline{F(z)}G(z) + \overline{F(\tau|z|)} \star G(\tau|z|) = \overline{F(\tau|z|)} \star \frac{G(\tau|z|)}{1 - \tau\frac{\bar{z}}{|z|}} + \frac{\overline{F(\tau|z|)}}{1 - \tau\frac{z}{|z|}} \star G(\tau|z|). \tag{3}$$

Proof. We use x and u as modulus and angle variables of z, $z = x\,\mathrm{e}^{\mathrm{i}u}$. We have

$$\overline{F(z)}G(z) = \sum_{r\geqslant 0} \overline{f_r} x^r\,\mathrm{e}^{-\mathrm{i}ru} \sum_{s\geqslant 0} g_s x^s\,\mathrm{e}^{\mathrm{i}su} = \sum_{q\in\mathcal{Z}} c_q(x)\,\mathrm{e}^{\mathrm{i}qu}$$

with

$$c_q(x) = \sum_{s-r=q} x^{r+s} \overline{f_r} g_s.$$

Consequently

$$c_{q>0}(x) = \sum_{r\geqslant 0} x^{2r+q}\overline{f_r}g_{r+q} = G(\tau)\star\sum_{r\geqslant 0} x^{2r}\overline{f_r}x^q\tau^{r+q}$$

$$c_{q>0}(x) = G(\tau)\star\overline{F(\tau x^2)}(x\tau)^q$$

similarly

$$c_{q<0}(x) = \sum_{s\geqslant 0} x^{2s-q}\overline{f_{s-q}}g_s = \overline{F(\tau)}\star G(\tau x^2)(x\tau)^{-q}$$

and

$$c_0(x) = \overline{F(\tau x)}\star G(\tau x).$$

Then we get

$$\overline{F(z)}G(z) = \sum_{q\geqslant 0}[(x\tau)^q G(\tau)\star\overline{F(\tau x^2)}\,\mathrm{e}^{\mathrm{i}qu} + (x\tau)^{-q}\,\mathrm{e}^{-\mathrm{i}qu}\overline{F(\tau)}\star G(\tau x^2)] - \overline{F(\tau x)}\star G(\tau x)$$

$$= G(\tau)\star\frac{\overline{F(\tau x^2)}}{1-\tau z} + \overline{F(\tau)}\star\frac{G(\tau x^2)}{1-\tau\bar{z}} - \overline{F(\tau x)}\star G(\tau x)$$

which completes the proof thanks to lemma 1. □

Corollary 1. *Let W be an entire series with a unit radius. Denoting* $\|S(\tau)\|^2 = \overline{S(\tau)}\star S(\tau)$, *the identity*

$$|W(z)|^2 + \|W(\tau|z|)\|^2 = 2\,\mathrm{Re}\left(\overline{W(\tau|z|)}\star\frac{W(\tau|z|)}{1-\tau\frac{\bar{z}}{|z|}}\right) \tag{4}$$

holds.

Proof. This is a particular case of proposition (2) with $F = G = W$. □

Corollary 2. *Let W be an entire real series of radius 1.* $|z| < 1$ *then*

$$|W(z)|^2 = W(\tau|z|)\star W(\tau|z|)\frac{1-\tau^2}{\left|1-\tau\frac{z}{|z|}\right|^2}. \tag{5}$$

Proof.

$$|W(z)|^2 = 2\,\mathrm{Re}\left(\overline{W(\tau|z|)}\star\frac{W(\tau|z|)}{1-\tau\frac{\bar{z}}{|z|}}\right) - \|W(\tau|z|)\|^2$$

$$= W(\tau|z|)\star W(\tau|z|)\left(\frac{1}{1-\tau\frac{\bar{z}}{|z|}} + \frac{1}{1-\tau\frac{z}{|z|}} - 1\right) = W(\tau|z|)\star W(\tau|z|)\frac{1-\tau^2}{\left|1-\tau\frac{z}{|z|}\right|^2}.$$

□

Proposition 3. *F, G are two series such that* $\rho_F\rho_G > 1$. *If z verifies* $\frac{1}{\rho_G} < |z| < \rho_F$, *then*

$$F(z)G\left(\frac{1}{z}\right) + F(\tau)\star G(\tau) = F(\tau)\star\frac{G(\tau)}{1-\tau z} + \frac{F(\tau)}{1-\frac{\tau}{z}}\star G(\tau). \tag{6}$$

Proof.

$$F(z)G\left(\frac{1}{z}\right) = \sum_{r\geqslant 0} f_r z^r\sum_{s\geqslant 0} g_s z^{-s} = \sum_{q\in\mathcal{Z}} z^q\sum_{r-s=q} f_r g_s = \sum_{q\in\mathcal{Z}} d_q z^q$$

where

$$d_q = \sum_{r-s=q} w_r w_s.$$

Using once more the representation of w_r

$$d_{q\geqslant 0} = F(\tau) \star \tau^q G(\tau)$$

and

$$d_{q\leqslant 0} = G(\tau) \star \tau^{-q} F(\tau)$$

we get

$$F(z)G\left(\frac{1}{z}\right) = F(\tau) \star \frac{G(\tau)}{1-\tau z} + G(\tau) \star \frac{F(\tau)}{1-\frac{\tau}{z}} - F(\tau) \star G(\tau). \qquad \square$$

Corollary 3. *Let W be an entire series with radius* $\rho_W > 1$ *and* $\frac{1}{\rho_W} < |z| < \rho_W$. *Then*

$$W(z)W\left(\frac{1}{z}\right) = W(\tau) \star W(\tau) \frac{(1-\tau^2)}{(1-\tau z)\left(1-\frac{\tau}{z}\right)}. \tag{7}$$

Proof. By replacing F and G by W at proposition 3 and rearranging the resulting sum, we obtain (7). $\square$

Remark 2. The previous representations are exceptional in their simplicity and compactness (even for the simple $W^2(z)$, attempts to build relevant formulae failed). They are optimal in a sense that will emerge later (at least for what they have been built for).

2. Mean on the regular unit *n*-gon

Now, we wish to use the previous formulae to compute the mean of complex functions on the regular *n*-gon.

Let $f(z)$ be a function defined on an n-symmetric subset of $\mathcal{C}$ (i.e. stable through product by $\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$); we denote

$$\{f\}_n(z) = \frac{1}{n}\sum_{j=1}^{n} f(z\omega_j)$$

its mean on the regular unit n-gon ($\sim$ group of the nth roots of unit) where $\omega_j = \mathrm{e}^{\mathrm{i}\frac{2\pi j}{n}}$.

Lemma 2. *Let* $p = an + p'$ *be the Euclidean division of p by n, then we have*

$$\left\{\frac{\bar{z}^p}{1-vz}\right\}_n = \frac{\bar{z}^p (vz)^{p'}}{1-v^n z^n}$$

$$\left\{\frac{z^p}{1-vz} - z^p\right\}_n = \frac{z^p (vz)^{n-p'}}{1-v^n z^n}.$$

Proof. As a matter of fact, it is sufficient to show the result on the set $|vz| < 1$ (using the field of rational fractions) where the Taylor expansion at 0 holds

$$\left\{\frac{\bar{z}^p}{1-vz}\right\}_n = \bar{z}^{an}\left\{\sum_{k\geqslant 0} v^k z^k \bar{z}^{p'}\right\}_n = \bar{z}^{an}\sum_{q\geqslant 0} v^{nq+p'} z^{nq+p'} \bar{z}^{p'} = \bar{z}^p \frac{(vz)^{p'}}{1-v^n z^n}. \qquad \square$$

A similar proof can be made for the other identity.

Proposition 4. *Under the hypotheses of proposition 2, we have*

$$\{z^p\overline{F(z)}G(z)\}_n(z) = z^p\overline{F(\tau|z|)} \star \frac{G(\tau|z|)\left(\tau\frac{\bar{z}}{|z|}\right)^{p'}}{1-\tau^n\frac{\bar{z}^n}{|z|^n}} + z^p\frac{\overline{F(\tau|z|)}\left(\tau\frac{z}{|z|}\right)^{n-p'}}{1-\tau^n\frac{z^n}{|z|^n}} \star G(\tau|z|).$$

Corollary 4. *Under the hypotheses of corollary 1, we have*

$$\{z^p|W(z)|^2\}_n(z) = z^p\overline{W(\tau|z|)} \star \frac{W(\tau|z|)\left(\tau\frac{\bar{z}}{|z|}\right)^{p'}}{1-\tau^n\frac{\bar{z}^n}{|z|^n}} + z^p\frac{\overline{W(\tau|z|)}\left(\tau\frac{z}{|z|}\right)^{n-p'}}{1-\tau^n\frac{z^n}{|z|^n}} \star W(\tau|z|).$$

Corollary 5. *Under the hypotheses of corollary 2 and with* $p' = \left[n\mathrm{frac}\left(\frac{p}{n}\right)\right]$ *and* $a = \left[\frac{p}{n}\right]$, *we have*

$$\{z^p|W(z)|^2\}_n(z) = |z|^p\,\mathrm{e}^{\mathrm{i}anu}W(\tau|z|) \star W(\tau|z|)\frac{\tau^{p'} - \tau^{2n-p'} + \mathrm{e}^{\mathrm{i}p'u}[\tau^{n-p'} - \tau^{n+p'}]}{\left|1-\tau^n\frac{z^n}{|z|^n}\right|^2}.$$

Proof. It is sufficient to apply the operator $\{\}_n$ where $\arg(z)$ occurs and after using lemma 2 we get the three previous corollaries. □

The three important practical cases are the following:

Corollary 6. *Let* W *be real with a radius greater or equal to 1. Assume* z *verifies* $|z| < 1$ *and* n *is a natural integer. Then we have*

$$\{|W(z)|^2\}_n(z) = W(\tau|z|) \star W(\tau|z|)\frac{1-\tau^{2n}}{\left|1-\tau^n\frac{z^n}{|z|^n}\right|^2} \tag{8}$$

$$\{z^p|W|^2\}_n(|z|) = |z|^pW(\tau|z|) \star W(\tau|z|)\frac{\tau^{p'}+\tau^{n-p'}}{1-\tau^n} \tag{9}$$

$$\{z^p|W|^2\}_n\left(|z|\mathrm{e}^{\mathrm{i}\frac{\pi}{n}}\right) = (-1)^{[\frac{p}{n}]}|z|^pW(\tau|z|) \star W(\tau|z|)\frac{\tau^{p'}-\tau^{n-p'}}{1+\tau^n}. \tag{10}$$

Proof. The previous identities are particular cases of corollary 5. □

Proposition 5. *Under the hypotheses of proposition (3), we have*

$$\left\{F(z)G\left(\frac{1}{z}\right)\right\}_n(z) + F(\tau) \star G(\tau) = F(\tau) \star \frac{G(\tau)}{1-\tau^nz^n} + \frac{F(\tau|z|)}{1-\frac{\tau^n}{z^n}} \star G(\tau). \tag{11}$$

Corollary 7. *Under the hypotheses of corollary 3, we have*

$$\left\{W(z)W\left(\frac{1}{z}\right)\right\}_n(z) = W(\tau) \star W(\tau)\frac{(1-\tau^{2n})}{(1-\tau^nz^n)\left(1-\frac{\tau^n}{z^n}\right)}. \tag{12}$$

Now, to complete the construction of the previous formalism, we need an intrinsic expression for $W \star S = \sum_{k\geqslant 0} w_ks_k$, i.e. which does not involve coordinates s_k of $S(\tau)$ in the canonical basis; this is possible under certain restrictive conditions on W, and this is the goal of the following sections.

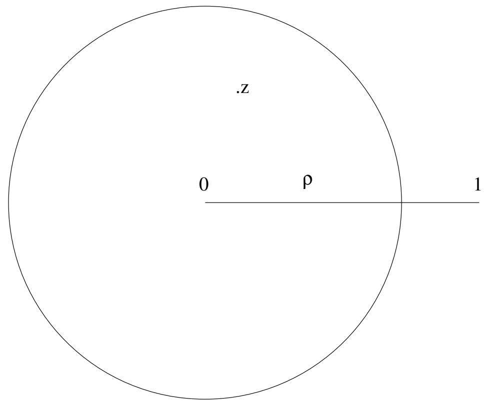

Figure 1. Circle contour for Cauchy's formula.

3. Intrinsic expression for $W\star$

In this section, we still assume that W is an entire series with radius 1.

z is a complex number, ρ a real number, such that

$$|z| < \rho < 1.$$

The sense of integration is always counterclockwise.

Let us recall Cauchy's formula on the next oriented contour (see figure 1):

$$W(z) = \frac{1}{2\mathrm{i}\pi}\oint_{|\xi|=\rho}\frac{W(\xi)}{\xi - z}\,\mathrm{d}\xi.$$

We compute in this integral the change of variable:

$$\tau = \frac{1}{\xi} \qquad \mathrm{d}\xi = -\frac{1}{\tau^2}\,\mathrm{d}\tau \qquad W(z) = \frac{1}{2\mathrm{i}\pi}\oint_{|\tau|=\frac{1}{\rho}}\frac{W\left(\frac{1}{\tau}\right)}{\tau}\frac{\mathrm{d}\tau}{1-\tau z}.$$

This leads to the intrinsic expression for $W\star$:

$$W\star S = \frac{1}{2\mathrm{i}\pi}\oint_{|\tau|=\frac{1}{\rho}}\frac{W\left(\frac{1}{\tau}\right)}{\tau}S(\tau)\,\mathrm{d}\tau$$

and so we get the fundamental identity

$$w_k = W\star\tau^k = \frac{1}{2\mathrm{i}\pi}\oint_{|\tau|=\frac{1}{\rho}}\frac{W\left(\frac{1}{\tau}\right)}{\tau}\tau^k\,\mathrm{d}\tau.$$

Remark 3. This representation is truly intrinsic with respect to W, but the integration contour lies on the complex field which is not endowed with the sign; this is why we apply more restrictive hypotheses so as to integrate in the real field.

Proposition 6. *We assume the following:*

(1) W is entire with radius 1 at the neighbourhood of the origin and possesses an holomorphic extension on $\mathcal{C} - [1; +\infty[$;

(2) For every $a > 1$, *the following limits exist:*

$$\lim_{\epsilon\to 0} W(a + \mathrm{i}|\epsilon|) = W^+(a)$$

$$\lim_{\epsilon\to 0} W(a - \mathrm{i}|\epsilon|) = W^-(a);$$

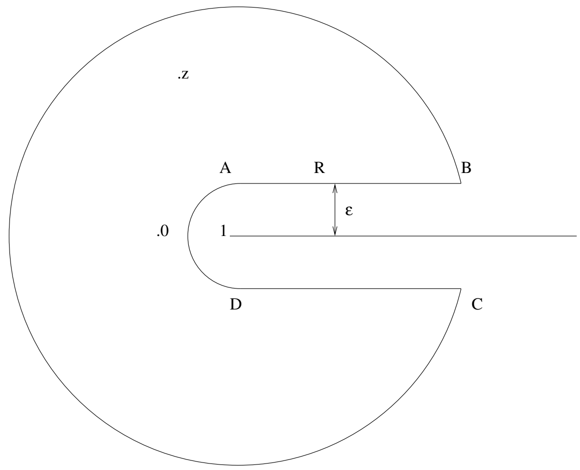

Figure 2. Deformed contour for Cauchy's formula.

(3)

$$|W(a+\mathrm{i}\epsilon)| \leqslant aG(a)$$

when ϵ is close to 0, where $G \in L^1(]1;+\infty[)$,

(4)

$$\lim_{|z|\to+\infty} |W(z)| = 0$$
$$\lim_{z\to 1} |(z-1)W(z)| = 0$$

then we have

$$W \star S = \frac{1}{2\mathrm{i}\pi}\int_0^1 \frac{W^+\left(\frac{1}{\tau}\right) - W^-\left(\frac{1}{\tau}\right)}{\tau} S(\tau)\,\mathrm{d}\tau.$$

Proof. We use the previous contour of integration and study the behaviour of the integral when $\epsilon \to 0$ and $R \to +\infty$ (see figure 2).

(1) On the path DA, using hypothesis 4, we have

$$\left|\int_{\mathrm{DA}} \frac{W(\xi)}{\xi - z}\,\mathrm{d}\xi\right| = \left|\int_{3\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{W(1+\epsilon\,\mathrm{e}^{\mathrm{i}u})}{1+\epsilon\,\mathrm{e}^{\mathrm{i}u} - z}\epsilon\,\mathrm{e}^{\mathrm{i}u}\,\mathrm{d}u\right| \to 0$$

when $\epsilon \to 0$.

(2) On the path BC, similarly

$$\left|\int_{\mathrm{BC}} \frac{W(\xi)}{\xi - z}\,\mathrm{d}\xi\right| = \left|\int_{\arg(B)}^{\arg(C)} \frac{W(R\,\mathrm{e}^{\mathrm{i}u})}{R\,\mathrm{e}^{\mathrm{i}u} - z} R\,\mathrm{e}^{\mathrm{i}u}\,\mathrm{d}u\right| \leqslant \text{const} \sup_{|z|=R} |W(z)| \to 0$$

as $R \to +\infty$.

(3) On the paths CD and AB:
Using the Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon\to 0}\left[\lim_{R\to\infty}\left(\int_{\mathrm{CD}} \frac{W(\xi)}{\xi - z}\,\mathrm{d}\xi + \int_{\mathrm{AB}} \frac{W(\xi)}{\xi - z}\,\mathrm{d}\xi\right)\right] = \int_1^{+\infty} \frac{W^+(a) - W^-(z)}{a - z}\,\mathrm{d}a.$$

Then, using $a = \frac{1}{\tau}$ the announced result is proved. □

Now we move on to the cases of homogenous potential (including the Newtonian one).

4. Application to homogeneous potential $1/r^{2\alpha}$

First, we assume that $0 < \alpha < 1$ and we apply the formulae of the previous sections for an homogeneous potential of the form $1/r^{2\alpha}$.

In this case we have

$$W_\alpha(z) = \frac{1}{(1-z)^\alpha} = \sum_{k\geqslant 0} w_k(\alpha) z^k$$

with

$$w_k(\alpha) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k+1)}.$$

It is possible to extend holomorphically W_α to $\mathcal{C}$-[1; +∞[using log, the holomorphic principal determination of the logarithm:

$$W_\alpha(z) = \exp(-\alpha \log(1-z)).$$

It is easy to verify that W_α fulfills the restrictive hypotheses of proposition 6.

Moreover, we can express both W_α^+ and W_α^-:

$$W_\alpha^+(a) = \exp(\mathrm{i}\pi\alpha)\frac{1}{(a-1)^\alpha}$$

$$W_\alpha^-(a) = \exp(-\mathrm{i}\pi\alpha)\frac{1}{(a-1)^\alpha}$$

with $a > 1$.

Then, from proposition 3 we get

$$W_\alpha \star S = \frac{1}{2\mathrm{i}\pi}\int_0^1 \frac{\frac{\exp(\mathrm{i}\pi\alpha)}{(\frac{1}{\tau}-1)^\alpha} - \frac{\exp(-\mathrm{i}\pi\alpha)}{(\frac{1}{\tau}-1)^\alpha}}{\tau} S(\tau)\,\mathrm{d}\tau = \frac{\sin(\pi\alpha)}{\pi}\int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^\alpha} S(\tau)\,\mathrm{d}\tau. \tag{13}$$

Hence, using corollary 6 (8) we get the explicit representations.

Proposition 7. *If* $0 \leqslant x < 1$, *then we have*

$$\frac{1}{n}\sum_{j=1}^{n} \frac{1}{\left(1+x^2-2x\cos\left(\frac{2\pi j}{n}+u\right)\right)^\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^\alpha}\frac{1}{(1-x^2\tau)^\alpha}\frac{1-(x\tau)^{2n}}{1+(x\tau)^{2n}-2(x\tau)^n\cos(nu)}\,\mathrm{d}\tau. \tag{14}$$

For values of x such that $x > 1$, *the homogeneity of the potential allows:*

Proposition 8. *For* $x > 1$

$$\frac{1}{n}\sum_{j=1}^{n} \frac{1}{\left(1+x^2-2x\cos\left(\frac{2\pi j}{n}+u\right)\right)^\alpha} = \frac{\sin(\pi\alpha)}{\pi}\int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^\alpha}\frac{1}{(x^2-\tau)^\alpha}\frac{x^{2n}-\tau^{2n}}{x^{2n}+\tau^{2n}-2x^n\tau^n\cos(nu)}\,\mathrm{d}\tau. \tag{15}$$

Remark 4. We gave an integral expression for the potential generated by n equal masses (their sum being equal to unity) disposed at the vertices of a regular n-gon in the plane containing this polygon.

For the general case, let $\alpha = q + \beta$, with $0 < \beta < 1$ and q integer. Then we have

$$w_k(\alpha) = w_k(q+\beta) = \frac{\Gamma(q+\beta+k)}{\Gamma(q+\beta)\Gamma(k+1)} = \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\Gamma(k+q+1)}{\Gamma(k+1)}\frac{\Gamma(q+k+\beta)}{\Gamma(\beta)\Gamma(q+k+1)}$$
$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)}\frac{\Gamma(k+q+1)}{\Gamma(k+1)} w_{q+k}(\beta)$$

but

$$\frac{\Gamma(k+q+1)}{\Gamma(k+1)}\tau^k = \left[\frac{\partial^q}{\partial\tau^q}(\tau^{q+k})\right]$$

then

$$W_{\beta+q} \star S = \frac{\Gamma(\beta)}{\Gamma(\alpha)} W_\beta \star \frac{\partial^q}{\partial\tau^q}(\tau^q S(\tau))$$

and, moreover,

$$\lim_{\beta\to 1} W_\beta \star S = \delta(S)$$

where

$$\delta(S) = S(1).$$

Thus, using previous formulae and proposition 6, we obtain:

Theorem 1. *Let $\alpha = q + \beta$, where q is a positive integer and $\beta \in]0, 1[$. Then, for every n, u real, and x real such that $(|x| < 1)$:*

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\left(1+x^2-2x\cos\left(\frac{2\pi j}{n}+u\right)\right)^\alpha} = \frac{1}{\Gamma(\alpha)\Gamma(1-\beta)}\int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^\beta}\frac{\partial^q}{\partial\tau^q}$$
$$\times\left(\tau^q\frac{1}{(1-x^2\tau)^\alpha}\frac{1-(x\tau)^{2n}}{1+(x\tau)^{2n}-2(x\tau)^n\cos(nu)}\right)\mathrm{d}\tau.$$

In the special case $\alpha = q + 1$, we get

$$\frac{1}{n}\sum_{j=1}^{n}\frac{1}{\left(1+x^2-2x\cos\left(\frac{2\pi j}{n}+u\right)\right)^\alpha}$$
$$= \left[\frac{1}{q!}\frac{\partial^q}{\partial\tau^q}\left(\tau^q\frac{1}{(1-x^2\tau)^\alpha}\frac{1-(x\tau)^{2n}}{1+(x\tau)^{2n}-2(x\tau)^n\cos(nu)}\right)\right]_{\tau=1}.$$

The following section emphasizes the efficiency of the previous representations in solving particular problems regarding relative equilibria.

5. Application of the representations to celestial mechanics

Consider a homogeneous potential of the kind $1/r^{2\alpha}$. Here, we assume that $0 < \alpha < 1$ but most of the results could be extended to larger intervals, which would lead to different formal calculus (Maple).

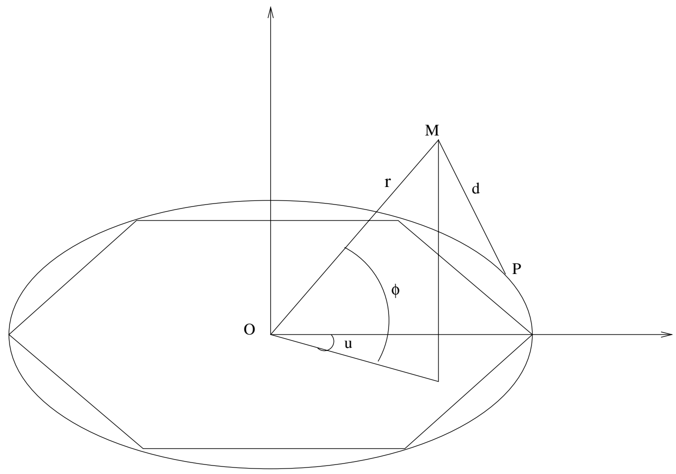

Figure 3. Spherical coordinates for M.

5.1. Field generated by a regular polygon

Let us consider n equal masses ($\frac{1}{n}$ for convenience) disposed at the vertices of a regular n-gon.

Theorem 2 (Lindow–Bang)**.** *The gravitational field generated at M by n equal masses disposed at the vertices of a regular n-gon is directed towards the centre of the polygon if and only if M belongs to an axis of symmetry of the polygon or to the axis passing by the centre of the polygon and orthogonal to its plane.*

Proof of the theorem. Without loss of generality, we suppose that the radius of the polygon is 1 (see figure 3). $\mathrm{e}^{-\mathrm{i}\psi}$ is the complex affix of a generic point P of the unit circle. M is a point of $\mathcal{R}^3$, (r, u, ϕ) being its spherical coordinates. To prove the theorem, it is sufficient to express in a correct way the conditions to make the orthoradial components of the field vanish, which allows us $r < 1$ (otherwise, we change r into $\frac{1}{r}$, the case $r = 1$ being a limit case). Everything is clear if $\cos(\phi) = 0$, so, we assume $\cos(\phi) > 0$.

$$
\begin{aligned}
d^2 = \|PM\|^2 &= (r\sin(\phi))^2 + (r\cos(\phi)\cos(u) - \cos(\psi))^2 + (r\cos(\phi)\sin(u) + \sin(\psi))^2 \\
&= 1 + r^2 - 2r\cos(\phi)\cos(u + \psi).
\end{aligned}
$$

The potential generated by the previous system at M takes the form

$$
V_n = \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\left(1 + x^2 - 2x\cos(\phi)\cos\left(\frac{2\pi j}{n} + u\right)\right)^{\alpha}}.
$$

Then we can find $r' \in]0; 1[$ such that

$$
\frac{2r'}{1 + (r')^2} = \frac{2r\cos(\phi)}{1 + r^2}
$$

because the second member lies between 0 and 1.

So we have

$$V_n = \left(\frac{1+(r')^2}{1+r^2}\right)^{\alpha} \frac{1}{n}\sum_{j=1}^{n} \frac{1}{\left(1+(r')^2 - 2r'\cos\left(\frac{2\pi j}{n}+u\right)\right)^{\alpha}}$$

$$E_u = -\frac{\partial V_n}{\partial u} = n\sin(nu)\left(\frac{1+(r')^2}{1+r^2}\right)^{\alpha} \frac{\sin(\pi\alpha)}{\pi}$$
$$\times \int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^{\alpha}} \frac{2(r'\tau)^n}{(1-\tau(r')^2)^{\alpha}} \frac{1-(r'\tau)^{2n}}{(1+(r'\tau)^{2n} - 2(r'\tau)^n\cos(nu))^2}\,\mathrm{d}\tau.$$

Clearly, this expression vanishes together with $\sin(nu)$.

$$E_\phi = -\frac{\partial V_n}{\partial \phi} = 2r\alpha\sin(\phi)\left(\frac{1+(r')^2}{1+r^2}\right)^{\alpha} \frac{1}{n}\sum_{j=1}^{n} \frac{\cos\left(u+\frac{2\pi j}{n}\right)}{\left(1+(r')^2 - 2r'\cos\left(\frac{2\pi j}{n}+u\right)\right)^{\alpha+1}}. \qquad \square$$

Then, the conclusion is a consequence of the following lemma:

Lemma 3. *Let* $\alpha \in]0;1[$ *and* $r' \in]0;1[$. *Then,*

$$\frac{1}{n}\sum_{j=1}^{n} \frac{\cos\left(u+\frac{2\pi j}{n}\right)}{\left(1+(r')^2 - 2r'\cos\left(\frac{2\pi j}{n}+u\right)\right)^{\alpha+1}}$$

does not vanish.

The proof of this technical lemma is in the appendix.

In conclusion, the necessary and sufficient condition for the field to be central at M is

$$\cos(\phi) = 0$$

or

$$\sin(nu) = 0 \qquad \text{and} \qquad \sin(\phi) = 0$$

which is, in other words, the desired result.

With less cost, we can announce

Theorem 3. *Considering a potential of the form* $\frac{1}{r^{2\alpha}}$ $(0 < \alpha \leqslant 1)$ *and a configuration composed by p homothetic n-gons, masses at the vertices of the ith being equal to* $\frac{m_i}{n}$, *the field generated by this system at M is central iff M lies on an axis of symmetry of the configuration or on the axis orthogonal to it and passing through its centre.*

The proof is similar to the previous one, one having just to take into account a sum of terms of the same sign.

Remark 5. This result of centrality is still true for values of α belonging to larger intervals, but we failed to prove it for every value of α, the differential term in the expression of the potential making the number of terms to take into account grow exponentially. We can always prove that, when this result is true for $\alpha = q$, it is still true for $\alpha \in]0; q]$.

The following results are a corollary in words of relative equilibria.

Let us recall that a relative equilibrium is a solution of the n-body problem such that the mutual distances of the bodies remain constant, each body rotating about the centre of masses of the system with the same angular velocity.

5.2. Two polygons in relative equilibrium

Proposition 9. *Let Π_1 (resp. Π_2) be a regular n-gon centred around a mass m_0 at O, m_1 being at each of its vertices (resp. m_2). Then, m_0, Π_1 and Π_2 is a relative equilibrium iff they are homothetic or cursed with an angle equal to $\frac{\pi}{n}$ (and suitable ratio of radii).*

Remark 6. For more than two polygons, this result does not hold: there is a known numerical example in [5] of three equilateral triangles (two of them not being homothetic or $\frac{\pi}{n}$-cursed) in relative equilibrium. In this example, all masses are equal.

5.3. Another application of the transformation formulae of theorem 1

As an illustration of the previous formalism, we show some properties of $\gamma(n, \alpha)$, a determining quantity in some problems linked to relative equilibria (in [7] and for $\alpha = \frac{1}{2}$ the authors use asymptotic expansion of $\gamma(n, \frac{1}{2})$).

$$\gamma(n, \alpha) = \frac{1}{2^{2\alpha+1}n} \sum_{j=1,\cdot,n-1} \frac{1}{\left(\sin\left(j\frac{\pi}{n}\right)\right)^{2\alpha}}.$$

Proposition 10. *Let $0 < \alpha < 1$. $\gamma(n, \alpha)$ can be analytically extended to real positive values of n. Moreover, considered as a function of the continuous variable n, $\gamma(n, \alpha)$ is strictly increasing. In the particular Newtonian case $\alpha = \frac{1}{2}$, $\left(\gamma\left(n, \frac{1}{2}\right) - 1\right)$ vanishes only at a single value of n lying in $]472; 473[$, $\left(\gamma\left(n, \frac{1}{2}\right) - 1\right)$ being negative before this value and positive after.*

Proof. We denote

$$V(x, n, \alpha) = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{\left(1 + x^2 - 2x\cos\left(\frac{2\pi j}{n}\right)\right)^{\alpha}}$$

and

$$\gamma(x, n, \alpha) = \frac{1}{n} \sum_{j=1,\ldots,n-1} \frac{1}{\left(1 + x^2 - 2x\cos\left(\frac{2\pi j}{n}\right)\right)^{\alpha}} = V(x, n, \alpha) - \frac{1}{n} V(x, 1, \alpha). \qquad \square$$

Lemma 4. *For every $x \in]0; 1]$, $\gamma(x, n, \alpha)$ is analytic and increasing with respect to n.*

This lemma is shown in the appendix.

6. Conclusion

In this paper, we built a useful tool that helps to compute cyclic quantities of the complex variable, using formally representations based on a scalar product. This method permits some means to be contracted in an integral form of some rational functions, so as to reduce analytical properties to algebraic ones, especially when studying moduli of analytical functions and their means on regular polygons.

We showed, in a few examples linked to celestial mechanics, how this method can be applied to deduce important technical results. Among them, we proved that the field generated at a point M of the space R^3 by a regular n-gon with unit mass at each vertices, is central if and only if M is on an axis of symmetry of the polygon or on the perpendicular axis passing through its centre. We also showed that the configuration formed by two regular n-gons centred around a mass m_0 is a relative equilibrium if and only if the polygons are homothetic or cursed with the angle $\frac{\pi}{n}$ (and with a suitable ratio of radii).

Acknowledgment

We are indebted to A Albouy who pointed out to us the Lindow formula in the Newtonian case. We thank also A Chenciner for fruitful discussions.

Appendix

Proof of lemma 3. we use the integral representation of the potential (proposition 7) with the value $n = 1$:

$$\frac{1}{(1+r'^2-2r'\cos(u))^{\alpha}} = \frac{\sin(\alpha\pi)}{\pi}\int_0^1 \frac{\tau^{1-\alpha}}{[(1-\tau)(1-r'^2\tau)]^{\alpha}}\frac{1-(r'\tau)^2}{1+(r'\tau)^2-2r'\tau\cos(u)}\,\mathrm{d}\tau.$$

Then after derivation with respect to u, simplification by $\sin(u)$ and multiplication by $\cos(u)$:

$$\left\{\frac{r'\cos(u)}{(1+r'^2-2r'\cos(u))^{\alpha+1}}\right\}_n = \frac{\sin(\alpha\pi)}{\pi}\int_0^1 \frac{\tau^{1-\alpha}(1-(r'\tau)^2)}{[(1-\tau)(1-r'^2\tau)]^{\alpha}} \times\left\{\frac{r'\tau\cos(u)}{(1+(r'\tau)^2-2r'\tau\cos(u))^2}\right\}_n\mathrm{d}\tau.$$

It is sufficient to prove that

$$\left\{\frac{r'\tau\cos(u)}{(1+(r'\tau)^2-2r'\tau\cos(u))^2}\right\}_n = \mathrm{Re}\left(\left\{z'\left|\frac{1}{(1-z')^2}\right|^2\right\}_n\right)$$

does not change its sign (denoting $z' = \tau r'\,\mathrm{e}^{\mathrm{i}u}$).

Using corollary 5 with $p = p' = 1$, we get

$$\left\{z'\left|\frac{1}{(1-z')^2}\right|^2\right\}_n = |z'|W(\tau|z'|)\star W(\tau|z'|)\frac{\tau-\tau^{2n-1}+\mathrm{e}^{\mathrm{i}nu}(\tau^{n-1}-\tau^{n+1})}{1+\tau^{2n}-2\tau^n\cos(nu)}$$

and taking the real part (the homogeneity allows us to replace $r'\tau$ with r' without a loss of generality):

$$\frac{1}{n}\sum_{j=1}^{n}\frac{r'\cos\left(u+\frac{2\pi j}{n}\right)}{\left(1+r'^2-2r'\cos\left(\frac{2\pi j}{n}+u\right)\right)^2} = |z'|W(\tau|z'|)\star W(\tau|z'|)\frac{\tau-\tau^{2n-1}+\cos(nu)(\tau^{n-1}-\tau^{n+1})}{1+\tau^{2n}-2\tau^n\cos(nu)}.$$

Then, we use the intrinsic formula for $W\star$ when $W(\tau) = \frac{1}{(1-\tau)^2}$,

$$W(\tau)\star S(\tau) = \left[\frac{\partial}{\partial\tau}(\tau S(\tau))\right]_{\tau=1}$$

we get

$$\begin{aligned}\frac{1}{n}\sum_{j=1}^{n}\frac{r'\cos\left(u+\frac{2\pi j}{n}\right)}{\left(1+r'^2-2r'\cos\left(\frac{2\pi j}{n}+u\right)\right)} &= r'\left[\frac{\partial}{\partial\tau}\left(\frac{\tau}{(1-r'^2\tau)^2}\frac{r'\tau-(r'\tau)^{2n-1}+\cos(nu)((r'\tau)^{n-1}-(r'\tau)^{n+1})}{1+(r'\tau)^{2n}-2(r'\tau)^n\cos(nu)}\right)\right]_{\tau=1}\\ &= \frac{\cos(nu)f_1(r',n)+f_0(r',n)}{(1-r'^2)^3(1+r'^{2n}-2r'^n\cos(nu))^2}\end{aligned}$$

with

$$f_1(v,n) = v^{n-1}[4v^2(v^{2n}-1) + n(v^{2n}+1)(1-v^4)]$$

and

$$f_0(v,n) = 2v[(1-v^{2n})(1+v^{2n}) + nv^{2n-2}(v^4-1)].$$

To conclude, it is sufficient to show:

(i) $f_1(v,n) \geqslant 0$.
(ii) $f_0(v,n) - f_1(v,n) \geqslant 0$.

(i) is equivalent to

$$n(v^{2n}+1)(1-v^4) \geqslant 4v^2(1-v^{2n})$$

or

$$n\frac{v^{2n}+1}{1-v^{2n}} \geqslant \frac{4v^2}{1-v^4}.$$

Let $f(v,n) = n\frac{v^{2n}+1}{1-v^{2n}}$.

$$\frac{\partial f}{\partial n} = \frac{g(v^{2n})}{(1-v^{2n})^2}$$

with $g(y) = 1 + 2y\ln(y) - y^2$.

$$g'(y) = 2\ln(y) + 2 - 2y = 2(\ln(y) + 1 - y) \leqslant 0 \text{ so } g(y) \geqslant g(1) = 0.$$

So, f is increasing with respect to n and as $f(v,2) = \frac{2(1+v^4)}{1-v^4} \geqslant \frac{vu^2}{1-v^4}$, we get $f_1(r',n) \geqslant 0$.

For (ii), $f_0(v,n) - f_1(v,n)$ is of the same sign as

$$\begin{aligned}
&2(1-v^{2n})(1+v^{2n}) + 2nv^{2n-2}(v^4-1) - 4v^n(v^{2n}-1) - nv^{n-2}(v^{2n}+1)(1-v^4)\\
&\quad = 2(1-v^{2n})(1+v^n)^2 - n(1-v^4)(1+v^{2n}+2v^{2n-2})\\
&\quad \geqslant (1+v^n)^2(2(1-v^{2n}) - n(1-v^4))
\end{aligned}$$

because $1 + v^{2n} + 2v^{2n-2} \leqslant 1 + v^{2n} + 2v^n$ when $n \geqslant 2$.

But,

$$\frac{\partial^2}{\partial n^2}(2(1-v^{2n}) - n(1-v^4)) = -8v^{2n}\ln(v)^2 \leqslant 0$$

leads to increasing $2(1-v^{2n}) - n(1-v^4)$ with respect to n. The value of this quantity for $n = 2$ leads to its positivity when $n \geqslant 2$ and $v \in]0;1[$.

This shows that

$$\frac{1}{n}\sum_{j=1}^{n}\frac{\cos\left(u + \frac{2\pi j}{n}\right)}{\left(1 + r'^2 - 2r'\cos\left(\frac{2\pi j}{n} + u\right)\right)^{\alpha+1}}$$

can be rewritten as an integral of a positive continuous function, which ends the proof of lemma 3. □

Proof of lemma 4. We use the formula (2.14) with $u = 0$ to express $\gamma(x,n,\alpha)$:

$$\gamma(x,n,\alpha) = \frac{\sin(\pi\alpha)}{\pi}\int_0^1 \frac{\tau^{\alpha-1}}{(1-\tau)^\alpha}\frac{1}{(1-x^2\tau)^\alpha}\left(\frac{1+(x\tau)^n}{(1-(x\tau)^n)} - \frac{1}{n}\frac{1+(x\tau)}{(1-(x\tau))}\right)\mathrm{d}\tau.$$

Clearly, the right term of this identity makes sense for real positive values of n, which proves the analytical extension.

Increasing γ is the consequence of the increasing of the integrand term with respect to n. We get it while studying

$$f(v,n) = \frac{1+v^n}{1-v^n} - \frac{1}{n}\frac{1+v}{1-v}$$

$$\frac{\partial f}{\partial n} = \frac{g(v,n)}{n^2(1-v^n)^2(1-v)}$$

where

$$g(v,n) = 2n^2 v^n \ln(v)(1-v) + (1+v)(1-v^n)^2$$

$$\frac{\partial g}{\partial n} = 2v^n \ln(v) h(v,n)$$

with

$$h(v,n) = n^2 \ln(v)(1-v) - 1 + 2n - v - 2nv + v^n + v^{n+1}$$

$$\frac{\partial h}{\partial n} = 2(1+n\ln(v))(1-v) + v^n \ln(v)(1+v)$$

$$\frac{\partial^2 h}{\partial n^2} = \ln(v)(v^n \ln(v)(1+v) + 2(1-v)) \leqslant 0$$

but $\frac{\partial}{\partial v}(v^n \ln(v)(1+v) + 2(1-v)) = \ln(v)(v^n + n(1+v)v^{n-1}) + (1+v)v^{n-1} - 2 \leqslant 0$ and $v^n \ln(v)(1+v) + 2(1-v) = 0$ for $v = 1$. So, $\frac{\partial h}{\partial n}$ decreases with respect to n.

For $n = 1$, $\frac{\partial h}{\partial n}(v, 1) = 2(1+\ln(v))(1-v) + v\ln(v)(v+1)$ and we easily prove that this function is negative. So, $\frac{\partial h}{\partial n} \leqslant 0$ for all values of n. Then, $h(v,n)$ decreases with respect to n. But, $h(v,1) = (1-v)(\ln(v) + 1 - v) \leqslant 0$ which leads to $h(v,n) \leqslant 0$ and also to the fact that $g(v,n)$ increases with respect to n. Moreover, $g(v,0) = 0$ so we see that $g(v,n)$ is positive. The lemma is proved and also the proposition computing $x \to 1$. □

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